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Infinite-Energy Dyon-Like Solutions for Yang–Mills–Higgs Theory

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Two dyon-like solutions to the $SU(2)$ Yang–Mills–Higgs system are presented. These solutions are obtained from the BPS dyon solution by allowing the gauge fields to be complex, or by letting the free parameter of the new solution be imaginary. In both cases the measurable quantities connected with these new solutions are real. Although the new solutions are mathematically simple variations of the BPS solution, they have distinct characteristics.

1. THE DYON SOLUTIONS

In this paper two new infinite-energy dyon-like solutions to the $SU(2)$ Yang–Mills–Higgs equations are given. Although these solutions are simple mathematical variations of the well-known BPS solution (Prasad and Sommerfield, 1975; Bogomolnyi, 1976), they have different physical characteristics, and to our knowledge they have not appeared in the literature before.

The system studied in this paper is an $SU(2)$ gauge theory coupled to a scalar field in the triplet representation. The scalar field is taken to have no mass or self-interaction. The Lagrangian for this system is

$$\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \tfrac{1}{2}(D_\mu\phi_a)(D^\mu\phi^a) \tag{1}$$

where $G^a_{\mu\nu}$ is the field strength tensor of the $SU(2)$ gauge fields W^a_μ and D_μ is the covariant derivative of the scalar field. The equations of motion for this system are simplified through the use of a generalized Wu–Yang ansatz

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(Yang and Wu, 1968), which was used by Witten (1977) to study multiinstanton solutions

$$
\begin{aligned}
W_i^a &= \epsilon_{aij} \frac{r^j}{gr^2} [1 - K(r)] + \left(\frac{r_i r_a}{r^2} - \delta_{ia} \right) \frac{G(r)}{gr} \\
W_0^a &= \frac{r^a}{gr^2} J(r) \\
\phi^a &= \frac{r^a}{gr^2} H(r)
\end{aligned} \tag{2}
$$

$K(r)$, $G(r)$, $J(r)$, and $H(r)$ are the ansatz functions to be determined by the equations of motion. In terms of this ansatz the field equations of the Lagrangian in equation (1) reduce to the following set of coupled nonlinear equations.

$$
\begin{aligned}
r^2 K'' &= K(K^2 + G^2 + H^2 - J^2 - 1) \\
r^2 G'' &= G(K^2 + G^2 + H^2 - J^2 - 1) \\
r^2 J'' &= 2J(K^2 + G^2) \\
r^2 H'' &= 2H(K^2 + G^2)
\end{aligned} \tag{3}
$$

where the primes denote differentiation with respect to r. The solution to these equations, discovered by Prasad and Sommerfield (1975) and independently by Bogomolnyi (1976), is

$$
\begin{aligned}
K(r) &= \cos(\theta) Cr \operatorname{csch}(Cr), & G(r) &= \sin(\theta) Cr \operatorname{csch}(Cr) \\
J(r) &= \sinh(\gamma)[1 - Cr \coth(Cr)], & H(r) &= \cosh(\gamma)[1 - Cr \coth(Cr)]
\end{aligned} \tag{4}
$$

where C, θ, and γ are arbitrary constants. One of the nice properties of this solution is that it has finite field energy. In terms of the ansatz functions the energy density of the fields is

$$
\begin{aligned}
T^{00} = \frac{1}{g^2} \bigg(& K'^2 + G'^2 + \frac{(K^2 + G^2 - 1)^2}{2r^2} + \frac{J^2(K^2 + G^2)}{r^2} + \frac{(rJ' - J)^2}{2r^2} \\
& + \frac{H^2(K^2 + G^2)}{r^2} + \frac{(rH' - H)^2}{2r^2} \bigg)
\end{aligned} \tag{5}
$$

For the solution in equation (4) this gives a nonsingular energy density, which when integrated over all space yields a finite field energy of $E = 4\pi C \cosh^2(\gamma)/g^2$. This finite-energy property of the BPS solution is one of the main reasons for the interest in this classical solution.

To investigate the electromagnetic properties of such solutions 't Hooft (1974) defined a generalized gauge-invariant electromagnetic field strength tensor

$$F_{\mu\nu} = \partial_\mu(\hat{\phi}^a W^a_\nu) - \partial_\nu(\hat{\phi}^a W^a_\mu) - \frac{1}{g}\,\epsilon^{abc}\hat{\phi}^a(\partial_\mu\hat{\phi}^b)(\partial_\nu\hat{\phi}^c) \tag{6}$$

where $\hat{\phi}^a = \phi^a(\phi^b\phi^b)^{-1/2}$. This generalized $U(1)$ field strength tensor reduces to the usual expression for the field strength tensor if one performs a gauge transformation to the Abelian gauge where the scalar field only points in one direction in isospin space (i.e., $\phi^a = \delta^{3a}v$) (Arafune *et al.*, 1975). Thus the electric and magnetic fields of the BPS solution become

$$E_i = F_{i0} = \frac{r_i}{gr}\frac{d}{dr}\left(\frac{J(r)}{r}\right) = \frac{\sinh(\gamma)r_i}{gr^3}\,[C^2r^2\text{csch}^2(Cr) - 1]$$

$$B_i = \frac{1}{2}\,\epsilon_{ijk}F_{jk} = -\frac{r_i}{gr^3} \tag{7}$$

The magnetic field is that of a point monopole of strength $-4\pi/g$; the electric field is that of an extended charge configuration of charge $Q = -4\pi\sinh(\gamma)/g$. The electric charge density is

$$\begin{aligned}\rho(r) = \nabla\cdot\mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\,(r^2E_r)\\ &= \frac{2C^2\sinh(\gamma)\text{csch}^2(Cr)}{gr}\,[1 - Cr\coth(Cr)]\end{aligned} \tag{8}$$

Since the BPS solution has finite field energy, this has led to its interpretation as a magnetically and electrically charged particle, with the field energy interpreted as the mass of the particle. The ansatz functions $K(r)$, $G(r)$, $J(r)$, and $H(r)$ (and therefore the gauge and scalar fields) are real. If complex gauge fields and/or infinite-energy configurations are allowed, several more solutions can be found for equation (3). First, by looking at the complementary hyperbolic functions, one finds the following complex solution:

$$\begin{aligned}K(r) &= i\cos(\theta)Cr\,\text{sech}(Cr), & G(r) &= i\sin(\theta)Cr\,\text{sech}(Cr)\\ J(r) &= \sinh(\gamma)[1 - Cr\tanh(Cr)], & H(r) &= \cosh(\gamma)[1 - Cr\tanh(Cr)]\end{aligned} \tag{9}$$

Since the ansatz functions $K(r)$ and $G(r)$ are imaginary, the space components of the gauge fields will be complex. Despite this, all the above-listed physical quantities associated with this complex solution are real. Inserting the ansatz functions (9) into equation (5), we find that the field energy density is

$$T^{00} = \frac{2\cosh^2(\gamma)}{g^2}\left\{-C^2\,\mathrm{sech}^2(Cr)[1 - Cr\tanh(Cr)]^2 + \frac{[C^2r^2\,\mathrm{sech}^2(Cr) + 1]^2}{2r^2}\right\} \tag{10}$$

This energy density is real, but the total field energy is infinite due to the singularity at $r = 0$. Thus the above solution is more like a Wu–Yang monopole (Yang and Wu, 1968) or a charged point particle, as opposed to a finite-energy BPS dyon. Using equation (7), we find that the electric and magnetic fields associated with this solution are

$$E_i = \frac{-\sinh(\gamma)r_i}{gr^3}[C^2r^2\,\mathrm{sech}^2(Cr) + 1]$$

$$B_i = -\frac{r_i}{gr^3} \tag{11}$$

The complex solution has the same magnetic charge $(-4\pi/g)$ and the same electric charge $[-4\pi\sinh(\gamma)/g]$ as the BPS solution. By using equation (8), we find that the electric charge density for the complex solution is given by

$$\rho(r) = \frac{-2C^2\sinh(\gamma)\,\mathrm{sech}^2(Cr)}{gr}[1 - Cr\tanh(Cr)] \tag{12}$$

This charge density is real, has a singularity at the origin, and falls off exponentially for large r. Even though the space components of the gauge fields are complex, all the physical quantities calculated from it are real. The main difference between this solution and the BPS solution is the infinite field energy of the complex solution.

To obtain the next solution we apply the transformation $C \to iC$ to the complex solution of (9). This changes the hyperbolic functions into their trigonometric counterparts, and yields the following solution to (3):

$$K(r) = -\cos(\theta)Cr\sec(Cr), \qquad G(r) = -\sin(\theta)Cr\sec(Cr)$$

$$J(r) = \sinh(\gamma)[1 + Cr\tan(Cr)], \qquad H(r) = \cosh(\gamma)[1 + Cr\tan(Cr)] \tag{13}$$

This solution is completely real, unlike the complex hyperbolic solution (9). Even though this solution was obtained from the complex hyperbolic solution via a trivial transformation, it has very different features. Most obviously, the ansatz functions, and therefore the gauge and scalar fields, become singular when $Cr = n\pi/2$, where $n = 1, 3, 5, 7, \ldots$, and at $r = 0$. Thus this solution exhibits a series of concentric spherical surfaces on which its fields become singular as well as a point singularity at the origin. These singularities also show up in the energy density of this solution. Inserting the ansatz functions (13) in (5), we find that the energy density of this solution is

$$T^{00} = \frac{2\cosh^2(\gamma)}{g^2}\left[C^2\sec^2(Cr)[1 + Cr\tan(Cr)]^2 + \frac{[C^2r^2\sec^2(Cr) - 1]^2}{2r^2}\right] \tag{14}$$

The energy density becomes singular on the same spherical surfaces as the gauge and scalar fields. These spherical shells, on which the energy density becomes infinite, cause the total field energy of this solution to diverge. The electric and magnetic fields of this solution are obtained using (7),

$$E_i = \frac{-\sinh(\gamma)r_i}{gr^3}[1 - C^2r^2\sec^2(Cr)]$$

$$B_i = -\frac{r_i}{gr^3} \tag{15}$$

The magnetic field is the same as that of the BPS solution or the solution of (9). However, the electric field does not fall off for large r, but exhibits a somewhat periodic behavior due to the $\sec^2(Cr)$ term. Additionally, it becomes singular on the spherical shells given by $Cr = n\pi/2$ (with n = odd) and at $r = 0$. One could take this as an indication that the electric charge of this solution is located on these singular surfaces. Finally, the electric charge of this solution is infinite, as indicated by the electric field or by directly looking at the charge density

$$\rho(r) = \frac{2C^2\sinh(\gamma)\sec^2(Cr)}{gr}[1 + Cr\tan(Cr)] \tag{16}$$

Integrating this over all space yields an infinite electric charge. For the special case where $\gamma = 0$, one finds that the solution carries no electric charge, but only a magnetic charge. Even in this case the energy density becomes singular on the concentric spherical surfaces and at the origin. Both the BPS solution and the solution (9) have finite magnetic and electric charges. The solution (13), while having the same magnetic charge as the other two solutions, has an infinite electric charge in the general case when $\gamma \neq 0$. Although this solution is a dyon in the sense that it carries both magnetic and electric charge, it is probably not correct to view it as a particle-like solution. At this point it is unclear how one should view this solution. The point singularity at $r = 0$ and the spherical singular surfaces of this solution are similar to that of the Schwarzschild-like solutions presented in Singleton (1995, 1996). However, the solutions in Singleton (1995, 1996) only possessed one point singularity at the origin and/or one spherical surface singularity on which the fields and energy density diverged. One conjectured use for the singular Schwarzschild-like solutions was as a possible explanation of the confinement mechanism. When the Schwarzschild-like solution of Singleton (1995) is treated as a background field in which a test particle is placed, it is found that the spherical singularity acts as an impenetrable barrier which traps the test particle either in the interior or the exterior of the sphere (Singleton and Yoshida, 1995), giving a classical type of confinement. Similar results have been found for other singular solutions (Swank *et al.*, 1975; Lunev, 1993). In addition, Swank *et al.*, (1975) point out that such a classical type of confinement is only possible with infinite-energy solutions. Treating the present solution as a background field would also trap test particles between any two of the concentric spherical singularities. This trigonometric solution could possibly be used to solve the field equations in some limited range of r, and then it could be patched to one of the other solutions, which would solve the field equations for the remaining range of r. This is similar to what is sometimes done in general relativity, where one tries to patch an exterior solution for some matter distribution with some interior solution.

Finally, one can obtain a third solution to (3) by applying the transformation $C \to iC$ to the BPS solution (4). This yields

$$K(r) = \cos(\theta) Cr \csc(Cr), \qquad G(r) = \sin(\theta) Cr \csc(Cr)$$
$$J(r) = \sinh(\gamma)[1 - Cr \cot(Cr)], \qquad H(r) = \cosh(\gamma)[1 - Cr \cot(Cr)] \quad (17)$$

Unlike the solutions (9) and (13), this solution was briefly discussed by Hsu and Mac (1977) in their derivation of the BPS solution [i.e., Hsu and Mac start with a solution like that in (17) and apply the transformation $C \to iC$ to arrive at the BPS solution]. This solution is similar to that in (13), in that it replaces the hyperbolic functions of the original solution with their trigonometric counterparts. It should be noted that due to the linear Cr term in each solution, one can not obtain the solution (17) from the other trigonometric solution (13) by simply letting $Cr \to Cr - \pi/2$. Although these two trigonometric solutions are in this sense distinct (i.e., they are not simply related by the transformation $Cr \to Cr - \pi/2$), they are physically similar, since most of the comments concerning the solution (13) apply here as well. The singularities in the fields and energy density are now located on the spherical surfaces $Cr = n\pi$, where $n = 1, 2, 3, 4, \ldots$. The solution also has infinite total field energy, and infinite electric charge, unless $\gamma = 0$. As with all the other solutions, it possesses a magnetic charge of $-4\pi/g$.

Many of the physical characteristics of the solutions were substantially different in each case. However, the magnetic charge of all the solutions is the same. This comes about since the magnetic charge of each solution is a topological charge which carries the same value for each field configuration. The topological current k_μ is (Arafune *et al.*, 1975)

$$k_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\alpha\beta}\epsilon_{abc}\partial^\nu\hat{\phi}^a\partial^\alpha\hat{\phi}^b\partial^\beta\hat{\phi}^c \tag{18}$$

The topological charge of this field configuration is then

$$\begin{aligned} q = \int k_0 d^3x &= \frac{1}{8\pi}\int (\epsilon_{ijk}\epsilon_{abc}\partial^i\hat{\phi}^a\partial^j\hat{\phi}^b\partial^k\hat{\phi}^c)\, d^3x \\ &= \frac{1}{8\pi}\int \epsilon_{ijk}\epsilon_{abc}\partial^i(\hat{\phi}^a\partial^j\hat{\phi}^b\partial^k\hat{\phi}^c)\, d^3x \end{aligned} \tag{19}$$

For all the solutions one finds that $\hat{\phi}^a = r^a/r$, which is the same regardless of the ansatz function $H(r)$. In all cases we find that the topological charge is $q = 1$.

2. DISCUSSION AND CONCLUSIONS

In this paper we have presented two new exact solutions to the $SU(2)$ Yang–Mills–Higgs system. In the solution given by (9) we replaced the hyperbolic functions of the BPS solution with their complements and let the ansatz functions $K(r)$ and $G(r)$ be imaginary, thus making the space components of the gauge fields complex. The magnetic and electric fields of this solution indicate that it is a dyon carrying a magnetic charge of $-4\pi/g$ and an electric charge of $-4\pi \sinh(\gamma)/g$. Although this solution looks similar to the BPS solution, physically it may be more correct to think of it as a dyonic version of the Wu–Yang monopole, which also has divergent field energy due to a singularity at $r = 0$. One mathematically interesting feature of this solution is that all its physically measurable quantities are real despite having complex gauge fields.

The two other solutions presented here were obtained by replacing the hyperbolic functions with their trigonometric counterparts. Both of these solutions had completely real fields. The gauge and scalar fields developed singularities on an infinite series of concentric spherical shells. In addition, unless $\gamma = 0$, both solutions carry an infinite electric charge, which may be thought of as concentrated on the spherical shells. Although these solutions carry both magnetic and electric charge (unless $\gamma = 0$), it is difficult to see how they could be viewed as particle-like, and the name dyon is probably inappropriate [even in the case of the complex solution (9), which has infinite field energy, the fact that it has localized charges and a localized divergent energy density makes a particle-like interpretation somewhat reasonable].

The singular spherical surfaces of the trigonometric solutions are similar to the singular spherical surface of the Schwarzschild-like solution of Singleton (1995). The Schwarzschild-like solution, however, had only one singular spherical surface rather than an infinite series of concentric surfaces. The occurrence of these singular spherical surfaces in the trigonometric solutions as well as in the Schwarzschild-like solution of Singleton (1995) could be taken as an indication that such structures may be a common feature of classical solutions to the Yang–Mills–Higgs equations. At present it is not clear what interpretation can be given to these trigonometric solutions or what physical role, if any, they may play. One could argue that the singularities in the trigonometric solutions, and to a lesser extent the singularity in the solution (9), might indicate that they are not physically important. However, this is not necessarily the case, as can be seen by the example of the meron solution (De Alfaro *et al.*, 1976), which is singular, and yet is thought to play a role in some non-Abelian gauge theories (Callan *et al.*, 1978). One might consider using the trigonometric solutions to solve the field equations in some finite region around the origin, and patching them together with one of the other solutions which are better behaved as $r \to \infty$. In one sense the trigonometric solutions, despite their singularities, are interesting since, along with the Schwarzschild-like solutions of Singleton (1995, 1996), they may indicate that having spherical surfaces on which the fields diverge may be a common feature of some non-Abelian gauge theory solutions.

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